Construction of a Numeration Model: A Theoretical Analysis

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This paper analyses the major concepts and processes within decimal-number numeration in terms of the cognitions inherent in their structure to determine how they are related to each other, how they are different and, if possible, to determine which are more difficult conceptually. It synthesises this analysis into a cognitive model that provides a framework for decimal-number knowledge.

For my doctoral dissertation, I developed a test to assess students' understanding of place value and their ability to identify, regroup, count, order, and estimate decimal numbers (limited to tenths and hundredths). This test was administered after formal instruction was complete to determine what structural knowledge of decimal numbers the students had abstracted. In undertaking the structural analysis of the numeration processes, I drew heavily on the properties of the decimal number system and the following research.

In their analysis of the semantic features of whole numbers and decimal fractions, Resnick et al. (1989) argued that the two domains shared the semantic features of base, order, place value, and multiplicative structure (a property that is derived from the a result of the notion of base 10 embedded in the decimal number system). However, Resnick et al.'s list of semantic features did not provide the cognitive processes (e.g., determining number names, determining the size of a number, determining whether two numbers are the same or different in value, changing the name of a number without changing its value) that are required to process decimal numbers.

Children's ability to interpret the part/whole subconstruct of fractions is highly dependent on their notion of a unit (Steffe, 1986) and their ability to partition (Kieren, 1980; Lamon, 1996; Pothier & Sawada, 1983) and reconstruct units (Behr et al., 1992; Nik Pa, 1989; Steffe, 1986). Steffe (1986) identified four different ways of thinking about a unit, namely, *counting* (or *singleton) units*, *composite units*, *unit-of-units* and *measure unit,* with each type representing an increasing level of abstraction.

Behr et al. (1992) analysed the complexities involved in *unitising* (identifying singleton units) and *reunitising* (identifying composite units and unit-of-units) whole numbers and decimal numbers (which require invoking the measure unit to relate the part to the whole). (In this paper, the term "unitising" is meant to include the opposite process, namely, partitioning.) They claimed that the ability to reunitise (i.e., to change one's perception of the unit) requires a flexibility of thinking that may be beyond young children. This has importance for hundredths (and thousandths, etc) which need to be thought of as a number of hundredths sometimes and as a number of tenths at other times. Similarly, tenths need to be thought of as a number of tenths or as a number of hundredths.

To these notions of unit, I added a focus on the concomitant processes that are used to derive either "superunits" (i.e., larger numbers) or "subunits" (i.e., smaller numbers) (Baturo, 1998). I argued that, in decimal-number numeration, generating superunits requires an understanding of grouping units by powers of 10 whilst generating subunits requires an understanding of partitioning units by powers of 10. Halford (1993) claimed that transforming units and keeping track of the transformations requires a great deal of flexible thinking and increases cognitive load.

These research findings provided me with a cognitive scaffold for undertaking an analysis of the cognitions required for semantic knowledge (understanding) of the central concepts and processes of decimal-number numeration, namely, place value, number identification, regrouping, counting, ordering, and estimating. No cognisance was taken of syntactic knowledge (e.g., reunitising tenths as hundredths by inserting an "imaginary zero" after the tenths digit).

Structural Analysis

Cognitions required for place value. Understanding place value means associating place names with particular *positions* on a continuum, and knowing the *order* of the place names. For example, to write the standard number (i.e., 4.25) when given a set of jumbled places (e.g., 2 tenths, 5 hundredths, 4 ones) requires semantic knowledge of the place names, their position, and order – not a reliance on the syntactic "left to right" order in which the digits of a number are usually written. Understanding place value also means identifying those numbers whose values were unchanged by the insertion of zero in the leftmost or rightmost places (e.g., 03.78 and 3.780). This item requires a clear understanding of the *order* of the places so that internal insertions which alter the original place values can be recognised (as opposed to the zero insertion at the rightmost place of a decimal number which is the syntactic feature associated with renaming one decimal place as another). In their study of decimal-number computation, Hiebert and Wearne (1985) drew attention to the difficulties students have in knowing when zeros can be inserted without changing the value of the given number.

Like all other digits that comprise a given number, zero has a cardinal value as well as a positional value in relation to the number. Thus, in 74.05, the 0 means 0 tenths and indicates that, in the continued grouping of 7405 hundredths, there were no tenths left after grouping to make ones but there are 5 hundredths remaining. Therefore, from a place value aspect, 0 should be considered in the same way as other digits. However, as the identity element for addition, zero could be taken to mean "having no effect" because the addition (or subtraction) of 0 will not change the number to be operated on. Hence, 74.05 could be thought of as 70 + 4 + 0 + 0.05 or as 70 + 4 + 0.05 (employing the additive property of the decimal number system) showing that 0 tens has no effect in this situation and can be omitted from the calculation. However, students often translate "having no effect" as "zero means nothing (and can therefore be ignored)" (Baturo, 1998); overgeneralising of this rule can lead students to interpret 74.05 and 74.5 as having the same value. Thus, the use of zero in a number can invoke inappropriate access.

The "value" part of place value is dependent on the *base* and either grouping or partitioning the unit in terms of the base to make "superunits" or "subunits" respectively. Inherent in the processes of grouping and partitioning is size and a direction in which the change in size is represented. For example, grouping in terms of the base (10 in our system) makes a number 10 times larger which, in the decimal number system, is indicated by a movement of digits one place to the left. Conversely, partitioning in terms of the base makes a number 10 times smaller and this is shown symbolically by a shift of the digit/s one place to the right. Thus, a leftwards direction is associated with becoming larger in value whilst a rightwards shift is associated with becoming smaller. Therefore, place value is *continuous* (i.e., across whole-number and decimal-fraction places), *bi-directional* (to the left is 10 times larger in value – $\times 10$; to the right is 10 times smaller in value – $\div 10$), and *exponential* (i.e., nonadjacent places are related by powers of 10 – 10^2, 10^3, etc.)

In Baturo (1998), I defined the continuous, bi-directional, and exponential properties of the decimal number system as multiplicative structure. That is, if a given number is multiplied by powers of 10 (base), it becomes larger in value, not from a change of digits but from a change in position. Conversely, if a number is divided by powers of 10, it becomes smaller in value because of the change in position.

Cognitions Required for Number Identification

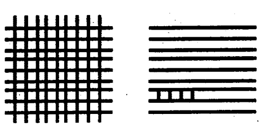
Numbers can be represented in pictorial, word, or digit form. When presented in digit form (e.g., 4.7, 6.39), the name of the endmost decimal-fraction place must be known in order to identify the numbers as tenths and hundredths respectively (i.e., *position* and *order*). When decimal numbers are presented in word form, the fraction name can be seen so it is merely a process of relating the fraction name to the appropriate fraction place (*position*). However, when presented in pictorial form (e.g., a prototypic 10 × 10 grid for hundredths), an ability to *unitise* (in conjunction with the measure unit) is required as well as a knowledge of position and order (to record the identified fraction correctly).

Cognitions Required for Regrouping

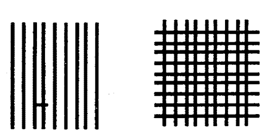
Numbers presented in pictorial, word or digit forms can be regrouped. Underlying the regrouping process is the notion of conservation of number which allows mathematicians flexibility in problem solving (e.g., knowing that 1 can be renamed as 10 tenths or 4 quarters, etc. without changing the value of the given number). Thus regrouping is one of the most powerful processes in mathematics. There are two main categories of regrouping tasks –those that require the given "unit" to be "superunitised" (i.e., transformed to a larger unit) and those that require the given unit to be "subunitised" (i.e., transformed to a smaller unit).

Solution paths (See Figure 1) show that, for each word/digit representation, the following cognitions are employed: (a) *additive structure* (e.g., 86 t = 80 t + 6 t); (b) *equivalence* (e.g., 80 t = 8 ones – superunitising; 1 one = 10 t – subunitising); (c) *position*; (d) *order*; and (e) *base*. However, when the number to be regrouped is in pictorial form, different cognitions are required, namely, *unitising* and *reunitising* (which requires the measure-unit). Figure 2 shows the complexity of thinking (drawing on Behr et al's, 1992, numeration) required for the simplest (Baturo, 1998) of these types, namely, forming subunits (e.g., renaming 2 tenths as 20 hundredths).

Superunitising		Subunitising	
86t 3h = __	7 t 14 h = __	2.09 = __ t __ h	0.52 = 4 t __h
86 t 3 h	7 t + 14 h	2.09	0.52
= 80t + 6t + 3h	= 7 t + 10 h + 4 h	= 2 ones + 9 h	= 5 t + 2 h
= 8 × 10t + 6t + 3h	= 7 t + 1 t + 4 h	= 2 × 1 one + 9 h	= 4 t + 1 t + 2 h
= 8 × 1 one + 6 t + 3 h	= 8 t + 4 h	= 2 × 10 t + 9 h	= 4 t + 10 h + 2 h
= 8 ones + 6 t + 3 h	= 0.84	= 20 t 9h	= 4 t 12 h
= 8.63			

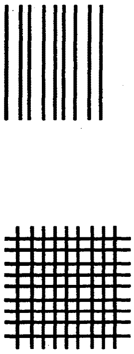


74 h = 70 h + 4 h
= 7 t 4 h



3 t 8 h = 30 h + 8 h
= 38 h

Figure 1. Classification of the regrouping items and their possible solution paths.



First: Unitise as tenths

10 × <u>1-unit</u> is unitised as 1 × <u>10-units</u> (which becomes the measure unit to which the shaded parts are related) so 2 is thought of as:

$2 \times {}^{1}/_{10}$ (1 × 10-units) or ${}^{2}/_{10}$ (1 × 10-units) = 0.2.

Second: Reunitise as hundredths

1 × <u>10-units</u> is reunitised as 1 × <u>10 × 10-units</u> by partitioning (mentally or physically) each tenth into 10 equal parts.

1 × <u>10 × 10-units</u> can then be reunitised as 1 × <u>100-units</u> (which becomes the new measure unit) so 2 becomes 20 and can be thought of as:

${}^{20}/_{100}$ (1 × 100-unit) or $20 \times {}^{1}/_{100}$ (1 × 100-unit) = 0.20

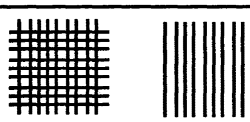
Figure 2. Cognitions involved in reunitising a pictorial representation of tenths as hundredths.

Cognitions Required for Counting

Counting forwards and backwards can be considered as the repeated addition or subtraction of an iterated unit. There are basically two types of tasks, namely, when the iterated unit is given (e.g., count in tenths from 6.7), or when the iterated unit has to be identified (e.g., continue the sequence, 6.5, 6.6, 6.7). I designated these types as Counting A and Counting B respectively. For Counting A types, a knowledge of *position* and *order* is required to identify the digit to which 1 tenth (or 1 hundredth, etc) has to be added and a knowledge of *order within places* (0<1<2<3...<9) is required to determine the direction (forwards/larger or backwards/smaller). In some cases (e.g., 6.9), the addition of 1 tenth will result in the need to *reunitise* so that the next number becomes 7 and not 6.10. For Counting B types, the iterated unit and the operation (+, −) needs to be determined. Determining the iterated unit requires a knowledge of *position* and *order across places* (i.e., tens>ones>tenths>hundredths) to decide which place/s remain constant and which place changes first.

Cognitions Required for Ordering

Ordering can be thought of as determining which of two or more given numbers is the smaller/est or larger/est in value so, basically, there is only one form of activity. The most common activities given to students are: (a) to select, from two given numbers, the number that has the larger or smaller value; and (b) to arrange three or four given numbers in ascending or descending order. Both types of activity involve comparing one number with another and require a knowledge that only like places (*position*) can be compared, combined with a knowledge of *order across places* (e.g., tens>ones>tenths>hundredths) and *order within places* (0<1<2<3...<9). However, pictorial comparisons of decimal fractions require perceptual as well as cognitive processes (see Figure 3).



The internal partitions of each shape need to be removed mentally to allow a focus on the relative size of the shading.

Figure 3. Comparing pictorial representations of unlike decimal fractions.

Cognitions Required for Estimating

This category subsumes approximating of which there are two types of activity (designated as Approximating A and Approximating B respectively), namely, to approximate a given decimal number to the nearest whole number (or tenth, etc.) or to identify a given number that has been approximated to the nearest whole number (or tenth, etc.). The A type is closed in that it can produce just one correct answer whilst the B type is open as there will be a range of numbers that will satisfy the condition. For the A types (e.g., round 6.2 to the nearest whole number), the following thinking may be employed.

> 6.2 has 6 ones and 2 tenths (*additive structure* – 6 ones + 2 t); 6.2 comes between 6 ones and 7 ones (*order within places* – identifying the parameters); 1 = 10 tenths (*equivalence*); $^{1}/_{2}$ = 5 tenths (*unitising* – related to *base*); 2 tenths is less than 5 tenths (*order within places*) so it is less than halfway between 6 ones and 7 ones; therefore 6.2 is closer to 6 ones than to 7 ones.

For the B types (e.g., what number could be approximated to 6), a knowledge of *order within places* is required to know that there are numbers that are both smaller and larger than 6. *Unitising* the whole as halves to determine the range of numbers on "either side" of the 6 is required in conjunction with *reunitisng* (halves as tenths) In effect, this type of approximation is the opposite process to that required for the ordering task in which the parameters of a range are given and selection of a number that falls within that range.

Whilst approximating activities produce exact answers, estimating activities usually produce answers that are inexact. When applied to pictorial representations, a knowledge of *order within places*, *equivalence* and *base* are required (see Figure 4).

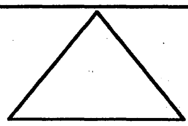
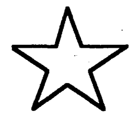
Colour 0.93 of this shape: 	Colour 0.6 of this shape: 
1 = 100 h (*equivalence*); 93 h is just less than 100 h *(order within places*) so colour just a bit less than the whole triangle	1 = 10 t (*equivalence*); $^{1}/_{2}$ = 5 t (*reunitising*); 6 t is just more than halfway *(order within places*) so colour just a bit more than half the star

Figure 4. Possible solution paths for nonprototypic estimation tasks.

Summary

Table 1 summarises the cognitions embedded in decimal-number numeration processes. It shows the structural complexity of regrouping and estimating, and the comparative structural simplicity of ordering.

Table 1
Cognitions Embedded in the Decimal-Number Numeration Processes

	Numeration Processes					
Cognition	PV	NI	RG	CO	OR	ES
Position	✓	✓	✓	✓	✓	✓
Base	✓	✓	✓	✓	✓	✓
Order	✓	✓	✓	✓	✓	✓
Equivalence	✓	–	✓	–	–	✓
Unitisation	–	✓	✓	✓	–	✓
Reunitisation	–	✓	✓	–	–	✓
Additive structure	–	–	✓	✓	–	✓
Multiplicative structure	✓	–	✓	–	–	–

Note. PV = place value; NI = number identification; RG = regrouping; CO = counting; OR = ordering; ES = estimating; – = not required.

Table 2 provides the test results from the six classes (two schools) involved in the main study of my PhD. The results generally support the hypotheses that regrouping and estimating are structurally complex whilst ordering is comparatively simple.

Table 2
Class Test Means (%) With Respect to the Decimal-Number Numeration Processes

	Numeration Processes					
Class	PV	NI	RG	CO	OR	ES
6A	39.7	68.5	39.8	63.4	73.6	57.9
6B	39.5	64.8	30.4	67.4	73.0	57.9
6C	48.1	69.6	45.8	74.9	71.6	74.6
6D	30.2	50.1	39.1	57.8	71.3	46.3
6E	43.2	62.4	30.5	57.0	58.8	44.8
6F	48.9	67.6	34.0	66.2	73.4	52.7

Constructing the Numeration Model

The test results showed that students' understanding of the cognitions embedded in the numeration processes varied considerably. For example, Table 2 revealed that place value knowledge was generally very poor. However, Table 3 shows that this was due mainly to the items related to multiplicative structure – students' understanding of position and order was generally reasonable. The disparity in means indicates that multiplicative structure is structurally more complex, and at a higher level, than position and order.

Table 3
Cognitions Embedded in the Place-Value Category of the Test

	Class					
Cognition	6A	6B	6C	6D	6E	6F
Position	79.3	76.9	81.5	61.6	72.4	76.9
Order	65.5	64.6	67.1	37.7	64.6	61.2
Multiplicative structure	21.8	15.8	25.6	20.7	27.3	36.5

As a result of similar analyses of the other numeration processes, I categorised the cognitions into the following three levels.

- Level 1 is associated with *position*, *base* and *order* and, as shown by Table 1, these cognitions are required for all processes. Therefore, without these cognitions, students would have no chance of processing decimal numbers. On the other hand, having Level 1 knowledge alone is not sufficient for a full understanding of all decimal-number numeration processes. For these reasons, I defined this level of cognitions as *baseline knowledge.*
- Level 2 is associated with unitising and equivalence. One or both of these cognitions appears to be required for Level 3. For this reason, I defined this level of cognitions as *linking knowledge.*
- Level 3 is associated with reunitising, additive structure and multiplicative structure. These cognitions appear to provide the superstructure for integrating the other levels and, for this reason, were defined as *structural knowledge.*

As a result of the foregoing analyses, I constructed a numeration model (Figure 5) to show the levels of cognitions and how they appear to be connected within and between the levels.

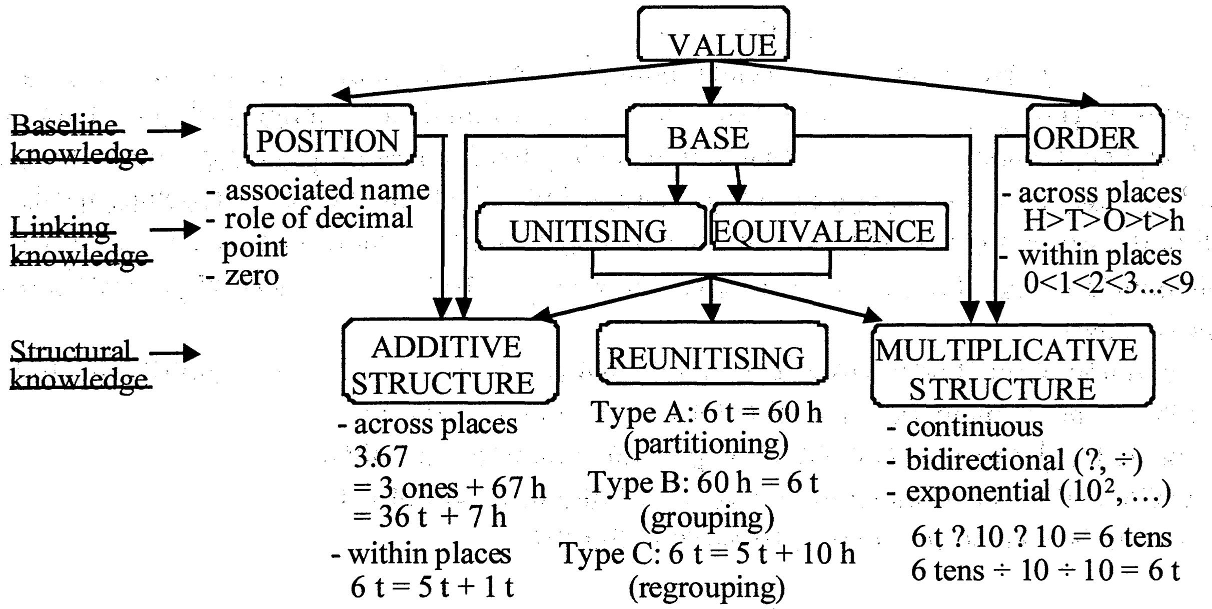


Figure 5. Cognitions and their connections embedded in the decimal number system.

The model portrays position, base (10) and order as the baseline knowledge from which all decimal-number numeration knowledge is derived. These are elementary cognitions that have a unary structure, that is, they consist of static memory-objects (Derry, 1996), for example, the names of the places and their positions. Inherent in the notion of position is knowledge of the associated place names, the role of the decimal point, and the effect of zero in a variety of places (external and internal). Zero plays an important role in the syntactic knowledge associated with abstract reunitising (i.e., inserting a zero after the tenths digit makes hundredths and, conversely, cutting off the 0 hundredths makes tenths), thinking which may or may not be associated with the semantic knowledge required for reunitising tenths as hundredths and hundredths as tenths.

Equivalence and unitising (Level 2 linking knowledge) are shown as emerging from the notion of base. These cognitions are dynamic in that one object is transformed to another through some operation. Therefore, they are binary in nature and, according to Halford's (1993) complexity metric, represent relational mappings.

Reunitising, additive structure and multiplicative structure (Level 3 structural knowledge) emerge from an amalgamation of Level 1 and Level 2 knowledge. They incorporate ternary relations, that is, they are transitive in nature (e.g., 6 tenths $\times$ 10 = 6 ones; 6 ones $\times$ 10 = 6 tens; $\therefore$6 tenths $\times$ 10 $\times$ 10 = 6 tens). Halford (1993) claimed that ternary relations are the basis of system mappings and consequently are much more complex cognitively than binary relations. To add to the complexity, some cognitions (order, additive structure and multiplicative structure) encompass different understandings – a consideration across places and within places (additive structure), different types of reunitising processes (Baturo, & Cooper, in press), and the continuous, bidirectional and exponential properties of multiplicative structure.

For the students in my study, additive structure tended to dominate multiplicative structure (e.g., associating +, – or ×, – instead of ×, ÷ relationships for growth to the left and right respectively). This may have been the result of overextended representation of numbers with base-10 blocks. With respect to reunitising, the students' results indicated that Type C regrouping was more difficult than the other two when decimal numbers in digit form were being considered. However, when decimal numbers were represented in pictorial form, Type B reunitising (i.e., grouping to form "superunits") was found to be more difficult. I believe that this phenomenon could be attributed to teaching that focuses on grouping whole numbers *but not decimal numbers* and partitioning fractions but not whole numbers. Teaching needs to provide the reverse experiences as well, namely, partitioning whole numbers to make other whole numbers (e.g., thousand to hundreds) and grouping fractions to make larger fractions or whole numbers.

The numeration model described in this paper is proposed as a signpost for future research in the domain of decimal-number numeration. As my test results showed, the students had major problems in processing decimal numbers and their teachers were unsure as to how decimal-number numeration processes could be taught effectively, indicating that there continues to be a need for vigorous research in this area.

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